Solving systems of linear equations - exact methods

Exact methods are those using which if working with exact numbers after a finite number of actions the result is an exact solution.

Problem formulation

A solution of the system of linear equations of the type $Ax = b$ is sought where

$$(1) \qquad A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & \ldots\ldots & & \\ a_{n1} & a_{n2} & & a_{nn} \end{pmatrix}, \qquad x = \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{pmatrix}, \qquad b = \begin{pmatrix} b_1 \\ b_2 \\ \ldots \\ b_n \end{pmatrix}.$$

In full form (1) looks like this:

$$\left| \begin{array}{lllll} a_{11}x_1 & +a_{12}x_2 & +\ldots & +a_{1n}x_n & =b_1 \\ a_{21}x_1 & +a_{22}x_2 & +\ldots & +a_{2n}x_n & =b_2 \\ & \ldots\ldots & & & \\ a_{n1}x_1 & +a_{n2}x_2 & +\ldots & +a_{nn}x_n & =b_n \end{array} \right. .$$

Gauss's method (Gauss's elimination)

One of the best-known methods for solving systems of linear algebra equations (SLAE) is **Gauss's method.** This method has different modifications and is usually used in two modes - with a chosen pivot element or without a chosen pivot element. Choosing a pivot (leading) element means that during the operation of division we will select the biggest possible divisor which would guarantee that we won't divide by zero or a number attending to zero. This means we won't lose accuracy. When solving big SLAE this mode of work is advisable.

Algorithm

With Gauss's method SLAE is transformed consecutively using exactly determined elementary transformations into equivalent systems and so the result is a triangular matrix which is not difficult to solve.

$$
(A\,|\,b)=\left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ & \dots\dots & & & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} & b_n \end{array}\right) \leftrightarrow \dots \leftrightarrow \left(\begin{array}{cccc|c} h_{11} & h_{12} & \dots & h_{1n} & d_1 \\ 0 & h_{22} & \dots & h_{2n} & d_2 \\ & \dots\dots & & & \dots \\ 0 & 0 & \dots & h_{nn} & d_n \end{array}\right)
$$

Example 1. Solve the system using Gauss's method with a chosen pivot element from a column:

$$
\left|\begin{array}{rrrrcr} 2x_1 & -x_2 & & +3x_4 & = & 9 \\ 4x_1 & -2x_2 & +5x_3 & & = & -10 \\ 3x_1 & +5x_2 & +2x_3 & -3x_4 & = & 0 \\ & -x_2 & +x_3 & -x_4 & = & -7 \end{array}\right.
$$

Solution: Elimination of x_1: From the first column we find the element with the highest absolute value, which is called pivot element. We mark it using a square. In our case this is the number 4. After that we change the places of the first row and the row with the pivot element which obviously does not change the solution of the system. The next elementary transformation is the division of the first row by the pivot element. Further on with elementary transformations we exclude x_1 from the second to the last row by multiplying the leading row respectively by (-2) and adding it to the second, by (-3) and adding with the third and so on. The aim is to get zeros in the first column.

Further on we continue analogically excluding x_2 with the sub-matrix without the first row and the first column: from the second row we find the element with the highest absolute value and so on up to the last unknown. This way we get a tridiagonal system which completes the straight move of the method.

$$
\left(\begin{array}{rrrr|r} 2 & -1 & 0 & 3 & 9 \\ \boxed{4} & -2 & 5 & 0 & -10 \\ 3 & 5 & 2 & -3 & 0 \\ 0 & -1 & 1 & -1 & -7 \end{array}\right) \leftrightarrow \left(\begin{array}{rrrr|r} \boxed{4} & -2 & 5 & 0 & -10 \\ 2 & -1 & 0 & 3 & 9 \\ 3 & 5 & 2 & -3 & 0 \\ 0 & -1 & 1 & -1 & -7 \end{array}\right) :(4) \leftrightarrow
$$

$$
\left(\begin{array}{rrrr|r} 1 & -\frac{1}{2} & \frac{5}{4} & 0 & -\frac{5}{2} \\ 2 & -1 & 0 & 3 & 9 \\ 3 & 5 & 2 & -3 & 0 \\ 0 & -1 & 1 & -1 & -7 \end{array}\right) .(-2)\ .(-3) \leftrightarrow \left(\begin{array}{rrrr|r} 1 & -\frac{1}{2} & \frac{5}{4} & 0 & -\frac{5}{2} \\ 0 & 0 & -\frac{5}{2} & 3 & 14 \\ 0 & \boxed{\frac{13}{2}} & -\frac{7}{4} & -3 & \frac{15}{2} \\ 0 & -1 & 1 & -1 & -7 \end{array}\right) \leftrightarrow
$$

$$\left(\begin{array}{cccc|c} 1 & -\frac{1}{2} & \frac{5}{4} & 0 & -\frac{5}{2} \\ 0 & \boxed{13/2} & -\frac{7}{4} & -3 & \frac{15}{2} \\ 0 & 0 & -\frac{5}{2} & -3 & 14 \\ 0 & -1 & 1 & -1 & -7 \end{array}\right) :(\tfrac{13}{2}) \quad \leftrightarrow \quad \left(\begin{array}{cccc|c} 1 & -\frac{1}{2} & \frac{5}{4} & 0 & -\frac{5}{2} \\ 0 & 1 & -\frac{7}{26} & -\frac{6}{13} & \frac{15}{13} \\ 0 & 0 & -\frac{5}{2} & -3 & 14 \\ 0 & -1 & 1 & -1 & -7 \end{array}\right) \quad \leftrightarrow$$

$$\left(\begin{array}{cccc|c} 1 & -\frac{1}{2} & \frac{5}{4} & 0 & -\frac{5}{2} \\ 0 & 1 & -\frac{7}{26} & -\frac{6}{13} & \frac{15}{13} \\ 0 & 0 & \boxed{-5/2} & 3 & 14 \\ 0 & 0 & \frac{19}{26} & -\frac{19}{13} & -\frac{76}{13} \end{array}\right) :(-\tfrac{5}{2}) \quad \leftrightarrow \quad \left(\begin{array}{cccc|c} 1 & -\frac{1}{2} & \frac{5}{4} & 0 & -\frac{5}{2} \\ 0 & 1 & -\frac{7}{26} & -\frac{6}{13} & \frac{15}{13} \\ 0 & 0 & 1 & -\frac{6}{5} & -\frac{28}{5} \\ 0 & 0 & \frac{19}{26} & -\frac{19}{13} & -\frac{76}{13} \end{array}\right) .(-\tfrac{19}{26}) \quad \leftrightarrow$$

$$\left(\begin{array}{cccc|c} 1 & -\frac{1}{2} & \frac{5}{4} & 0 & -\frac{5}{2} \\ 0 & 1 & -\frac{7}{26} & -\frac{6}{13} & \frac{15}{13} \\ 0 & 0 & 1 & -\frac{6}{5} & -\frac{28}{5} \\ 0 & 0 & 0 & \boxed{-\frac{38}{65}} & -\frac{114}{65} \end{array}\right) :(-\tfrac{38}{65}) \quad \leftrightarrow \quad \left(\begin{array}{cccc|c} 1 & -\frac{1}{2} & \frac{5}{4} & 0 & -\frac{5}{2} \\ 0 & 1 & -\frac{7}{26} & -\frac{6}{13} & \frac{15}{13} \\ 0 & 0 & 1 & -\frac{6}{5} & -\frac{28}{5} \\ 0 & 0 & 0 & 1 & 3 \end{array}\right)$$

Reverse move: The resulting “triangular” system is equivalent to the given. It is solved from the “bottom-up”.

From the last equation we find x_4:

$$\rightarrow \quad \boxed{x_4 = 3}\ .$$

From the last but one equation we find x_3:

$$x_3 - \frac{6}{5}x_4 = -\frac{28}{5} \quad \rightarrow \quad x_3 - \frac{6}{5}.3 = \frac{28}{5} \quad \rightarrow \quad \boxed{x_3 = -2}\ .$$

From the second equation we find x_2:

$$x_2 - \frac{7}{26}x_3 - \frac{6}{13}x_4 = \frac{15}{13} \quad \rightarrow \quad x_2 - \frac{7}{26}.(-2) - \frac{6}{13}.3 = \frac{15}{13} \quad \rightarrow \quad \boxed{x_2 = 2}\,.$$

And in the end from the first equation we find x_1:

$$x_1 - \frac{1}{2}x_2 + \frac{5}{4}x_3 + 0.x_4 = -\frac{5}{2} \quad \rightarrow \quad x_1 - \frac{1}{2}.2 + \frac{5}{4}.(-2) + 0.3 = -\frac{5}{2} \quad \rightarrow \quad \boxed{x_1 = 1}\ .$$

Answer: $x = (1;\ 2;\ -2;\ 3)$.

In this example pivot elements are: $4,\ \frac{13}{2},\ -\frac{5}{2},\ -\frac{38}{65}$.

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